Solution to Workshop 1 on Black Body Radiation

1. Write down the Black body intensity Formula

Those of you who wondered about which intensity formula to use here had a valid point but the one we want relates intensity to frequency as below*.

$$\boxed{I(\nu)=\frac{8\pi\nu^3}{c^2}\cdot\frac{h}{e^{h\nu/KT}-1}} \qquad (1)$$

2. Differentiate it with respect to the frequency and set equal to zero to find the maximum

This is quite a long and complicated step as you can all remember so let's first of all define two constants to make things easier to follow.

$$\text{Let; } A=\frac{8\pi h}{c^2}$$
$$\text{Let; } B=\frac{h}{KT} \qquad (2)$$

This means that equation 1 becomes...

$$I(\nu)=A\nu^3\cdot\frac{1}{e^{B\nu}-1} \qquad (3)$$

We need to differentiate this with respect to frequency, set it to zero and solve for ν in order to find the frequency at which the intensity is a maximum. There are two ways to do this, I'm going to do both.

Using the Product Rule

The function that we want to differentiate can be thought of as a product of two functions and differentiated using the product rule.

$$(fg)'=f'g+g'f \quad (4)$$

With $f=A\nu^3$ and therefore $f'=3A\nu^2$

and $g=\frac{1}{e^{B\nu}-1}$ and therefore $g'=\text{“}\mathit{difficult}\text{”}$

in order to calculate g' we need to first use a substitution,

$$\text{Let; } y=e^{B\nu}-1$$

Using the substitution we rewrite g in terms of y and use the chain rule method to find g'.

$g=\frac{1}{y}$ and therefore $\frac{dg}{dy}=-\frac{1}{y^2}$

$y=e^{Bv}-1$ and therefore $\frac{dg}{dv}=B\,e^{Bv}$

the chain rule tells us that g' is given by,

$$\frac{dg}{dv}=\frac{dg}{dy}\cdot\frac{dy}{dv}=-\frac{Be^{Bv}}{\left(e^{Bv}-1\right)^2}$$

putting everything into formula 4 gives us,

$$\boxed{\frac{dI}{dv}=\frac{3Av^2}{\left(e^{Bv}-1\right)}-\frac{Av^3\,B\,e^{Bv}}{\left(e^{Bv}-1\right)^2}} \qquad (6)$$

Using the Quotient Rule

I don't ever use the quotient rule because I always get the minus sign wrong and you can always use the product rule instead. In this case it is a bit easier though. It is as follows,

$$\left(\frac{f}{g}\right)'=\frac{f'g-g'f}{g^2} \quad (5)$$

With $f=A\nu^3$ and therefore $f'=3A\nu^2$

and $g=e^{B\nu}-1$ and therefore $g'=Be^{B\nu}$

putting everything into formula 5 gives us,

$$\frac{dI}{dv}=\frac{3Av^2\left(e^{Bv}-1\right)-Av^3\,B\,e^{Bv}}{\left(e^{Bv}-1\right)^2}$$

this is the same as

$$\frac{dI}{dv}=\frac{3Av^2\left(e^{Bv}-1\right)}{\left(e^{Bv}-1\right)^2}-\frac{Av^3\,B\,e^{Bv}}{\left(e^{Bv}-1\right)^2}$$

whereby the $\left(e^{Bv}-1\right)$ in the first term on the right cancels, giving,

$$\boxed{\frac{dI}{dv}=\frac{3Av^2}{\left(e^{Bv}-1\right)}-\frac{Av^3\,B\,e^{Bv}}{\left(e^{Bv}-1\right)^2}} \qquad (6)$$

Which is thankfully exactly the same as the answer we get using the product rule. It's also the correct answer, which is a bonus and why I've put it in a box.

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*You will sometimes see this written with different constants in front of it as on the “Planck's Law” page of wikipedia. This is completely equivalent to the formula here but the intensity is that emitted per unit area of the emitting body rather than per unit volume. Don't worry, if you ever use this in real life you'll be able to work out which to use and you'll be given the formula in an exam. It doesn't affect the answer or the Physics of the problem anyway because these constants cancel out anyway.

Okay, so by both methods of differentiating we've got the same answer, equation (6), but our job is only half done. Now we need to set this to zero and solve for ν. Setting to zero gives us

$$0=\frac{3Av^2}{\left(e^{Bv}-1\right)}-\frac{Av^3Be^{Bv}}{\left(e^{Bv}-1\right)^2}$$

moving one of the terms to the other side gives

$$\frac{3Av^2}{\left(e^{Bv}-1\right)}=\frac{Av^3Be^{Bv}}{\left(e^{Bv}-1\right)^2}$$

Cancelling an A, a v^2 and one $\left(e^{Bv}-1\right)$ gives us

$$3=Bv\cdot\frac{e^{Bv}}{\left(e^{Bv}-1\right)} \tag{7}$$

replacing the B we defined in equation 2 into equation 7 gives us the correct answer,

$$\boxed{3=\frac{hv}{KT}\frac{e^{hv/KT}}{\left(e^{hv/KT}-1\right)}}$$

3. You will obtain an equation that cannot be solved analytically. Use the high temperature approximation, where $e^x=1+x+...$ to rewrite the equation.

Okay, let's keep with equation 7 for the moment since it's still easier to deal with. The approximation we're asked to use is valid for small x because for small x, $x^3 \ll x^2 \ll x$. If you're not sure why it is valid or where it comes from you need to look up McLaurin series in any maths text book. These approximations are really really useful but I'm sure you know that already.

Here we're just going to substitue e^{Bv} with $1+Bv$ in the right hand side of equation 7. This gives us,

$$Bv\cdot\frac{e^{Bv}}{\left(e^{Bv}-1\right)}\approx\frac{Bv\cdot\left(1+Bv\right)}{1+Bv-1}=\left(1+Bv\right) \tag{8}$$

meaning we can approximate equation 7 as,

$$\boxed{3=\left(1+Bv\right)} \tag{9}$$

4. Solve the approximated equation to show that hv = 2KT. The actual (not approximate number) is 3.8. The result, hv = 3.8KT, is known as Wien's law.

Replacing the B we defined in equation 2 into equation 9 gives us the correct answer,

$$\boxed{hv=2KT}$$

Although the question says that the correct constant is 3.8 not 2 this is not completely true. 2.8 is nearer to the correct value and could explain comments by some of you that the Wien formula didn't always seem to give the correct answers of the peak frequency. For a more detailed discussion feel free to look right at the bottom of http://en.wikipedia.org/wiki/Wien_displacement_law_constant.

5. Integrate the formula for the intensity to obtain the total output, which is known as the Stefan-Boltzmann law. You may use the following integral:

$$\int_0^\infty \frac{x^3}{e^x-1}\cdot dx=\frac{\pi^4}{15} \tag{10}$$ [*]

A nice easy one to finish off with ;) No but seriously it's pretty easy. Do you remember equation 1 ?

$$I(\nu)=\frac{8\pi\nu^3}{c^2}\cdot\frac{h}{e^{h\nu/KT}-1} \tag{1}$$

Well, we just need to integrate this over all frequencies (ie from zero to infinity) to get the total output intensity. That is to say, we need to perform this horrible looking integration,

$$I_{total}=\frac{8\pi h}{c^2}\cdot\int_0^\infty\frac{\nu^3}{e^{h\nu/KT}-1}\cdot dv \tag{11}$$

* Just in case you're wondering, integrals like these are easily solved using a complicated bit of Maths called Complex Analysis and specifically the Cauchy Integral Theorem. It is a truly awesome bit of maths.

Thankfully we have the standard integral given to us so it won't be that hard. First we'll have to make a substitution so that the standard integral we're given works. The substitution must make the exponential part of the integral the same as in the standard integral and so we need to use,

$$x = h\nu / KT \tag{12}$$

Substituting equation 12 into equation 11 gives us,

$$I_{total} = \frac{8\pi}{c^2} \cdot \frac{K^3 T^3}{h^2} \int_0^\infty \frac{x^3}{e^x - 1} \cdot d\nu \tag{13}$$

The final thing we need to do is replace the dv with a dx in the integral. We do this by differentiating equation 12 to find the relation between dx and dv*.

$$\frac{dx}{d\nu} = h / KT \quad \text{and therefore} \quad d\nu = \frac{KT}{h} dx \tag{14}$$

substituting equation 14 into equation 13 gives us an equation containing the standard integral we know and a lot of other constants.

$$I_{total} = \frac{8\pi}{c^2} \cdot \frac{K^4 T^4}{h^3} \int_0^\infty \frac{x^3}{e^x - 1} \cdot dx$$

Solving this equation leaves us with the Stefan-Boltzmann equation.

$$\boxed{I_{total} = \underbrace{\frac{8\pi^5 K^4}{15 h^3 c^2}}_{\sigma} \cdot T^4}$$

The End.

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* Never tell a Mathematician that this is what you did. Tell them you used the Chain Rule or something similar, but really this works fine.